Chair of Software Modeling and Verification (Computer Science 2)

RWTH AACHEN UNIVERSITY

A Framework for Online Optimization

Ongoing Research

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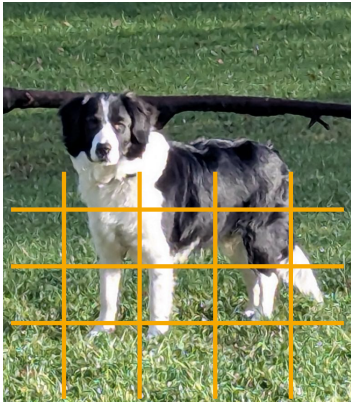
RWTH Aachen University
26.11.2025

Online Optimization

Short-Sighted Cow*

*Dog



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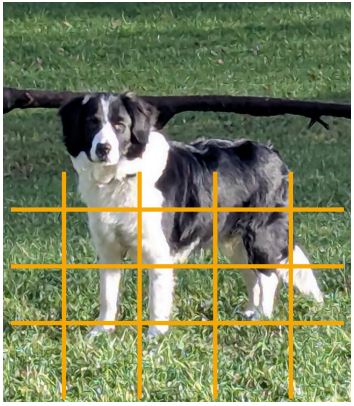
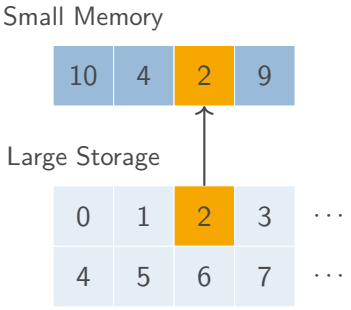


Paging Problem



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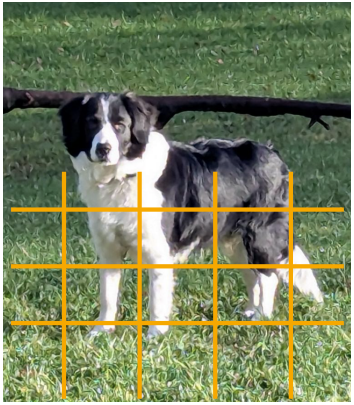

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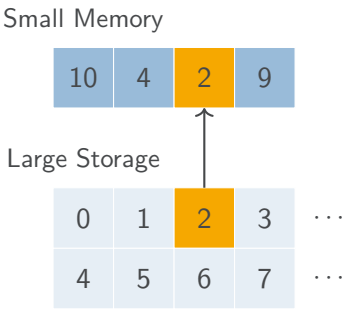

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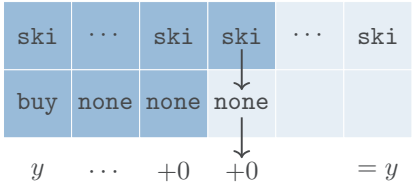

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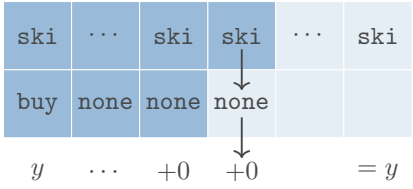

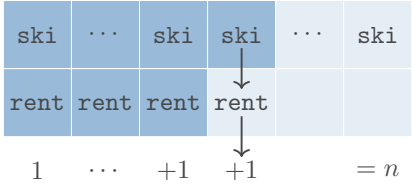

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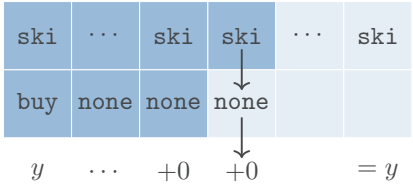

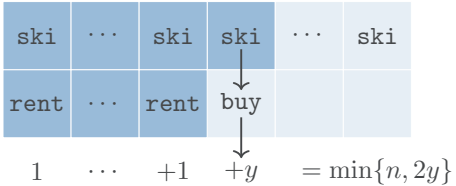

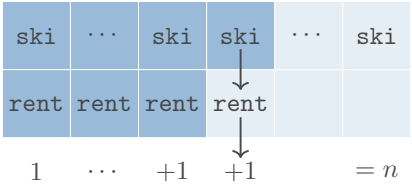

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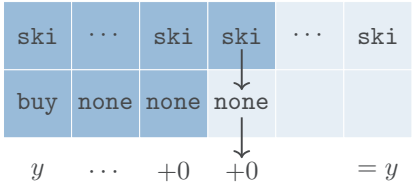
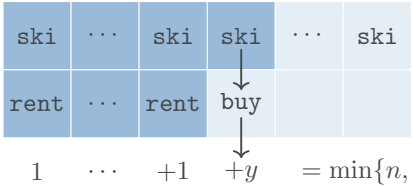
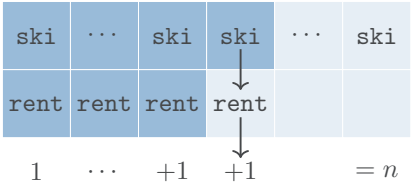
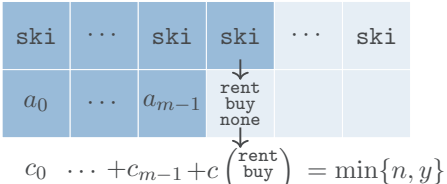


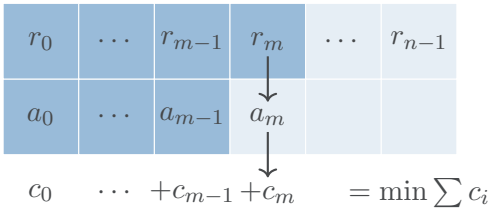






Online Optimization - Semantics

An online algorithm is one that receives a sequence of requests and, performs an immediate action in response to each request. Each sequence of requests and corresponding actions has an associated cost. *Richard M. Karp, 1992*



Online Optimization - Semantics

Optimization Problem

- requests $r_i \in \mathcal{R}$
- actions $a_i : \Sigma \to \Sigma \in \mathcal{A}$
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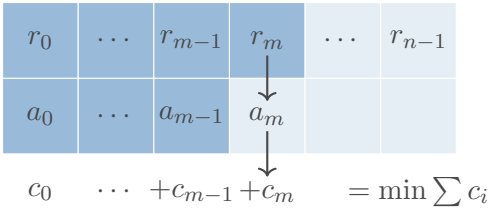

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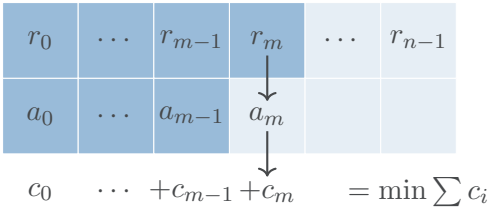

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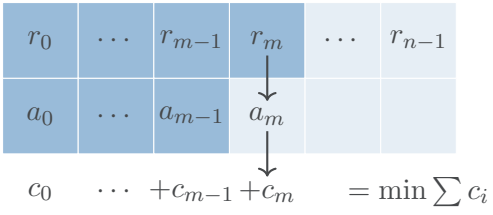

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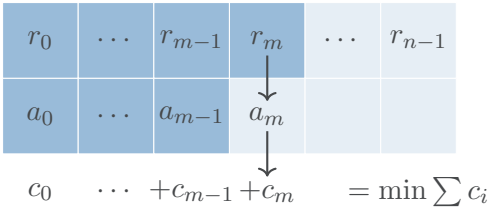

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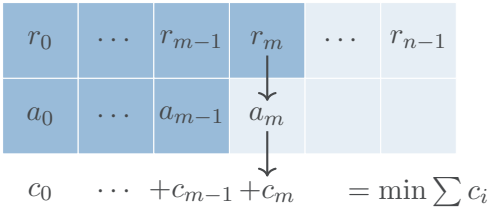

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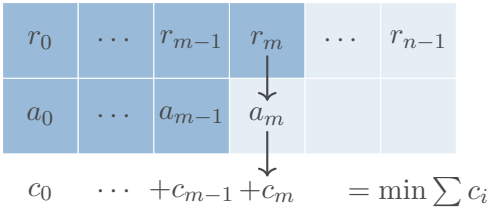

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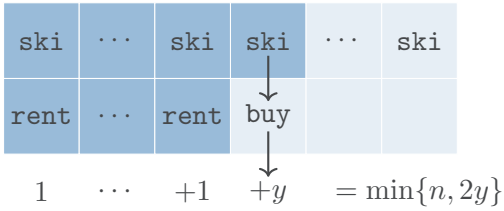
Ski-Rental Problem

Optimization Problem

- requests $\mathcal{R} = \{\texttt{ski}\}$
- actions $\mathcal{A} = \{\texttt{rent}, \texttt{buy}, \texttt{none}\}$
- costs $c_{\texttt{rent}} = 1$, $c_{\texttt{buy}} = y$, $c_{\texttt{none}} = [s = 1] \cdot 0$



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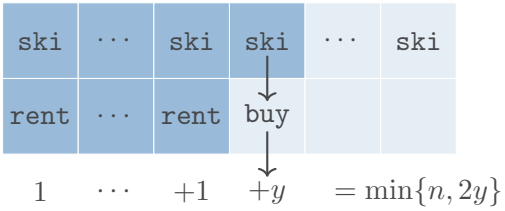

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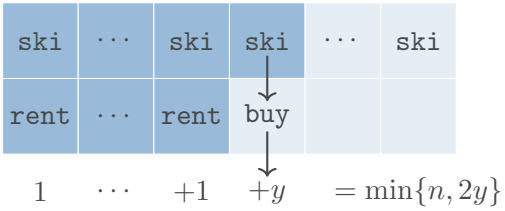

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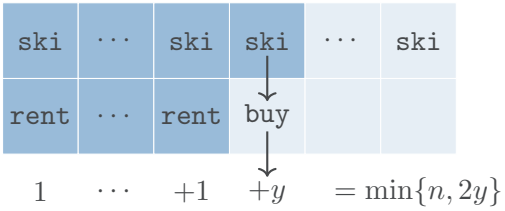

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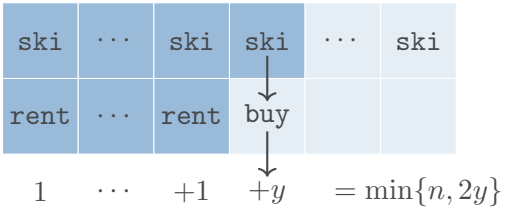

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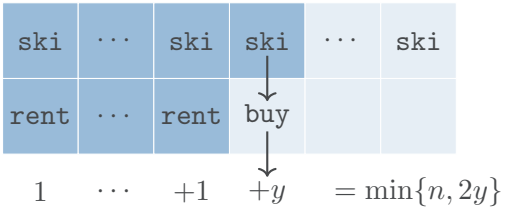

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Weighted Programming wGCL

$$
\begin{array}{rcl}
C & \to & x \coloneqq E \quad \mid \quad C_1 \mathbin{;} C_2 \quad \mid \quad \odot w \\
& \mid & \texttt{if}\,(\varphi)\,\{\,C_1\,\}\,\texttt{else}\,\{\,C_2\,\} \quad \mid \quad \texttt{while}\,(\varphi)\,\{C_1\} \quad \mid \quad \{\,C_1\,\} \oplus \{\,C_2\,\},
\end{array}
$$

where $x \in \texttt{Vars}$, E is an arithmetic expression, and $[\![w]\!] \in \mathbb{S}$

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Semiring $\mathbb{S} = (S, \oplus, \odot, \mathbb{0}, \mathbb{1})$ satisfies

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Omitted: ω-complete partial order, ω-continuity

Reasoning about Competitive Ratio

problem-specific proofs by hand

$$D := \sup_{R \in \mathcal{R}^n} \frac{c(R, \mathsf{Alg}(R, \epsilon))}{c(R, \mathsf{Alg}'(R, \epsilon))} = \sup_{R \in \mathcal{R}^n} \frac{?}{?}$$

framework for semi-automatic verification

Weakest Preweightings Semantics

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- parametrized over semiring $\mathbb{S}$
- undecidable in general
- semi-automatic verification tools (like Caesar) exist for some instances of weighted programming (e.g. for the probabilistic and the tropical semiring)

Ski-Rental Problem

Optimization Problem

- requests $\mathcal{R} = \{\texttt{ski}\}$
- actions $\mathcal{A} = \{\texttt{rent}, \texttt{buy}, \texttt{none}\}$
- costs $c_{\texttt{rent}} = 1$, $c_{\texttt{buy}} = y$, $c_{\texttt{none}} = [s = 1] \cdot 0$

Off $= \texttt{while}\ (m < n)\ \{$

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Ski-Rental Problem: Competitive Ratio

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$$\mathtt{wp}[\![\mathsf{Off}]\!](\mathfrak{1}) = n \oplus y = \min\{n, y\}$$

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$$D = \sup_{R \in \mathcal{R}^n} \frac{\sigma_R(n \oplus 2y)}{\sigma_R(n \oplus y)} = \sup_{n>0, y>1} \frac{\min\{n, 2y\}}{\min\{n, y\}} = 2$$

From Costs to Expected Costs

Consider the expectation semiring $\mathbb{S}_{\mathbb{E}} = ([0,1] \times \mathbb{R}^{\infty}_{\geq 0}, \oplus, \odot, (0,0), (1,0))$

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Often, randomized online algorithms perform better than deterministic online algorithms!

Ski-Rental Problem: Competitive Ratio of Randomized Algorithm

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&\quad m \coloneqq m + 1\\
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\end{aligned}
$$

$$
\begin{aligned}
&\mathtt{Rand} = \texttt{while}\ (m < n)\ \{\\
&\quad \texttt{if}\ (c < y) \to \{\, \odot (1 - p_m, 1 - p_m) \mathbin{;} c \coloneqq c + 1 \,\}\\
&\quad \oplus\ (c < y) \to \{\, \odot (p_m, p_m \cdot y) \mathbin{;} s \coloneqq 1 \mathbin{;} c \coloneqq y \,\}\\
&\quad \oplus\ (c \geq y) \to \{\, \odot [s = 0] \cdot (0, \infty) \mathbin{;} \texttt{skip} \,\}\\
&\quad m \coloneqq m + 1\\
&\}.
\end{aligned}
$$

Ski-Rental Problem: Competitive Ratio of Randomized Algorithm

$$
\begin{aligned}
&\mathsf{Off} = \texttt{while}\ (m < n)\ \{\\
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&\quad \oplus\ (\texttt{true}) \to \{\, \odot y \ \texttt{;}\ s \coloneqq 1 \,\}\\
&\quad \oplus\ (\texttt{true}) \to \{\, \odot [s = 1] \cdot 0 \ \texttt{;}\ \texttt{skip} \,\}\\
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&\}.
\end{aligned}
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$$\mathtt{wp}[\![\mathsf{Off}]\!](\mathfrak{1}) = n \oplus y = \min\{n, y\}$$

$$\mathtt{wp}[\![\mathsf{On}]\!](\mathfrak{1}) = E(n, y)$$

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$$

$$\mathtt{wp}[\![\mathsf{Off}]\!](\mathfrak{I}) = n \oplus y = \min\{n, y\}$$

$$
\begin{aligned}
&\mathtt{wp}[\![\mathsf{On}]\!](\mathfrak{I}) = E(n, y)\\
&= \textstyle\sum_{c=0}^{\min\{n,y\}-1} (y + c) \cdot \tilde{p}_c + \left(1 - \sum_{c=0}^{\min\{n,y\}-1} \tilde{p}_c\right) \cdot n.
\end{aligned}
$$

Ski-Rental Problem: Competitive Ratio of Randomized Algorithm

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$$

$$D = \sup_{R \in \mathcal{R}^n} \frac{\mathtt{wp}[\![\mathtt{Prog}]\!](\mathfrak{1})(\sigma_R)}{\mathtt{wp}[\![\mathtt{Off}]\!](\mathfrak{1})(\sigma_R)} = \sup_{n>0, y>1} \frac{E(n, y)}{\min\{n, y\}} = \frac{e}{e - 1} \approx 1.58$$

Using Weakest Preweightings for Competitive Ratio

Online Optimization:

- for each optimization problem manual proofs by hand

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Translation into Weighted Programming:

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Translation into Weighted Programming:

- uniform framework

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Translation into Weighted Programming:

- uniform framework
- semi-automatic verification via tools

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Translation into Weighted Programming:

- uniform framework
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- further research in this area applies directly

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Online Optimization:

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problem-specific proofs by hand

$$D := \sup_{R \in \mathcal{R}^n} \frac{c(R, \mathsf{Alg}(R, \epsilon))}{c(R, \mathsf{Alg}'(R, \epsilon))} = \sup_{R \in \mathcal{R}^n} \frac{\mathtt{wp}[\![\mathtt{Prog}]\!](\mathfrak{I})(\sigma_R)}{\mathtt{wp}[\![\mathtt{Prog}']\!](\mathfrak{I})(\sigma_R)}$$

framework for semi-automatic verification

Using Weakest Preweightings for Competitive Ratio

Type	Function	Program	Coherence
action $a_i \in \mathcal{A}$	$a_i : \Sigma \to \Sigma$	$\texttt{wGCL}(a_i)$	$\texttt{Obs}$
cost	$c_i : \Sigma \to \mathbb{S}$ with $A(\sigma_R) \mapsto c(R, A, a_i)$	$\texttt{WExp}(c_i)$	$\texttt{Obs}$
auxiliary action	$x_i : \Sigma \to \Sigma$	$\texttt{wGCL}(x_i)$	$\texttt{Aux}$
guard	$\varphi_i : \Sigma \to \{0, 1\}$	$\texttt{BExp}(\varphi_i)$	$\texttt{Obs} \cup \texttt{Aux}$

problem-specific proofs by hand

$$D := \sup_{R \in \mathcal{R}^n} \frac{c(R, \mathsf{Alg}(R, \epsilon))}{c(R, \mathsf{Alg}'(R, \epsilon))} = \sup_{R \in \mathcal{R}^n} \frac{\texttt{wp}[\![\texttt{Prog}]\!](\mathfrak{1})(\sigma_R)}{\texttt{wp}[\![\texttt{Prog}']\!](\mathfrak{1})(\sigma_R)}$$

framework for semi-automatic verification

Classical	Observable
Arithmetic Expressions AExp $E \to x \in \texttt{Vars} \mid n \in \mathbb{N} \mid E + E \mid E \cdot E \mid E \dot{-} E$	**Observable Arithmetic Expressions** AObs $E \to E_1 \mid \ldots \mid E_k \mid q \in \mathbb{N} \mid E + E \mid E \cdot E \mid E$ with $E_1, \ldots, E_k \in \texttt{AExp}$
Boolean Expressions BExp $\varphi \to E < E \mid \varphi \wedge \varphi \mid \neg\varphi$	**Observable Boolean Expressions** BObs $\varphi \to \varphi_1 \mid \ldots \mid \varphi_\ell \mid E < E \mid \varphi \wedge \varphi \mid \neg\varphi$ with $\varphi_1, \ldots, \varphi_\ell \in \texttt{BExp}$
Weighted Expressions WExp $w \to s \in \mathbb{S} \mid w \oplus w \mid w \odot w \mid E \cdot w \mid [\varphi] \cdot w$ with $a \in \texttt{AExp}$ and $\varphi \in \texttt{BExp}$	**Observable Weighted Expressions** WObs $w \to s \in \mathbb{S} \mid w \oplus w \mid w \odot w \mid E \cdot w \mid [\varphi] \cdot w$ with $E \in \texttt{AObs}$ and $\varphi \in \texttt{BObs}$
	$\texttt{BaseObs} = \{E_1, \ldots, E_k, \varphi_1, \ldots, \varphi_\ell\}$

References I


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