Weighted Rewriting: Semiring Semantics for Abstract Reduction Systems

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16.07.2025

Provenance Analysis in Databases

$$\psi = Ra \wedge (Pab \vee Pbb)$$

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a		true
b		false

P		
a	a	true
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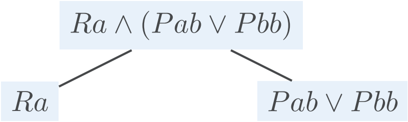

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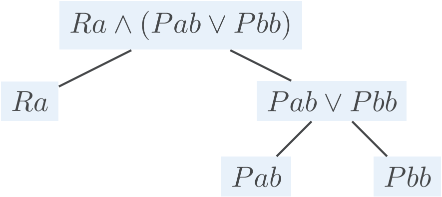

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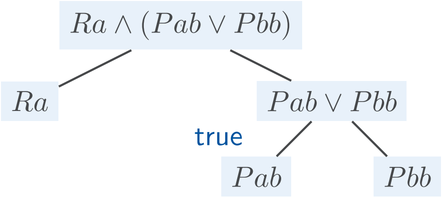

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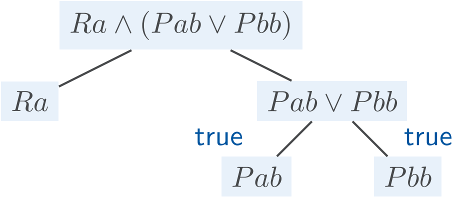

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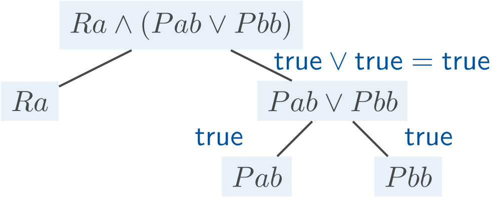

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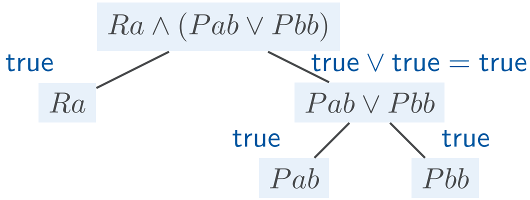

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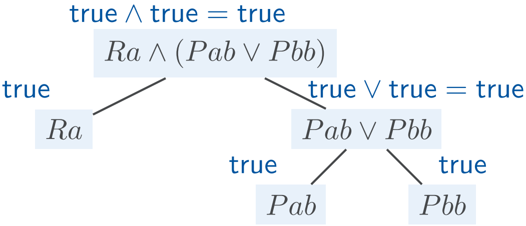

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P			cost
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a	b	true	7
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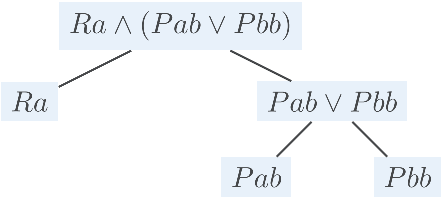

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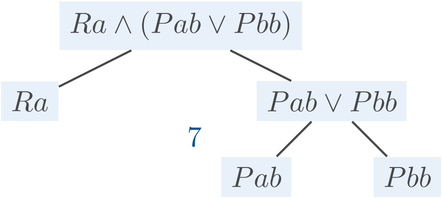

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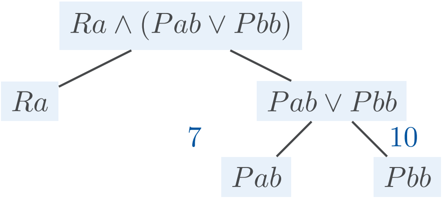

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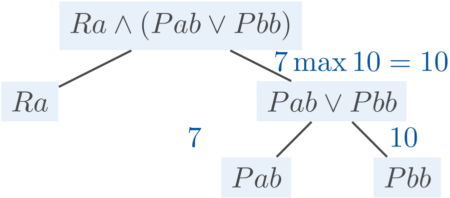

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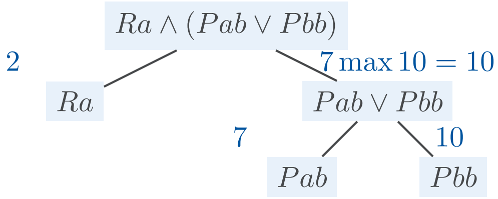

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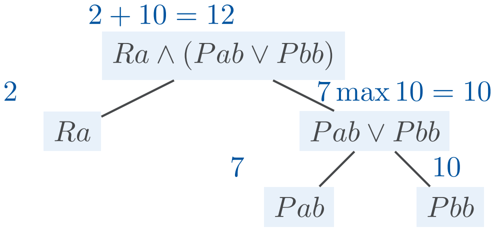

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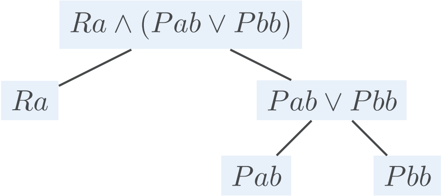

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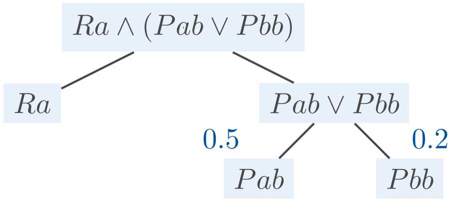

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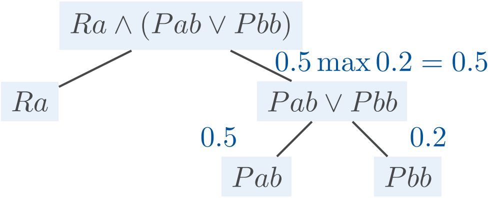

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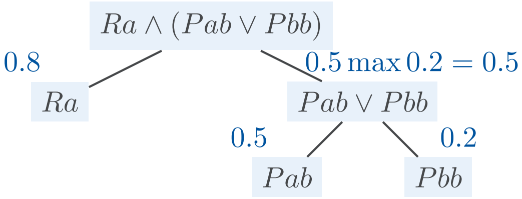

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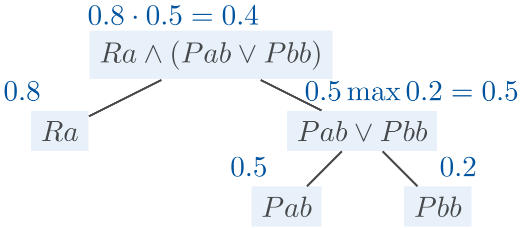

Goal

Abstract Reduction System

$$(A, \to)$$

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Provenance Analysis via Semirings

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Weighted Rewriting

Uniform Framework for Analyzing Abstract Reduction Systems

- Termination Analysis
- Safety Analysis
- Cost Analysis
- Complexity Analysis
- Arbitrary Combinations
- Many more!

Semiring

Semiring $\mathbb{S} = (S, \oplus, \odot, \mathbf{0}, \mathbf{1})$ with

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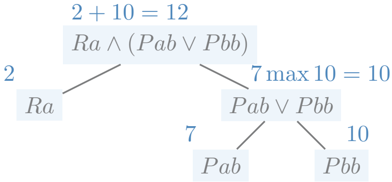
For example: $\mathbb{N} \subseteq \mathbb{N}^{\pm\infty}$ and $\sup \mathbb{N} = \top = \infty \in \mathbb{N}^{\pm\infty}$

One Framework Fits All

Weighted Abstract Reduction System $(A, \to, \mathbb{S}, \mathsf{f}_{\mathtt{NF}}, \mathsf{Aggr})$ with



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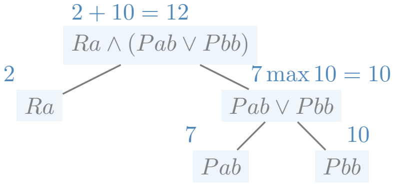
Costs in database:

Ra

$\psi \wedge \varphi \longrightarrow [\psi, \varphi]$

$\psi \vee \varphi \longrightarrow [\psi, \varphi]$



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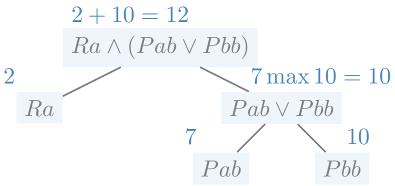

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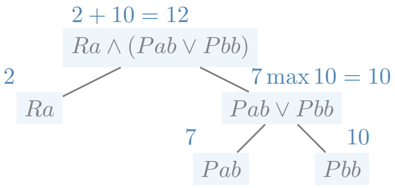

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Costs in database: Choose $\mathbb{S}_{\mathsf{arc}} = (\mathbb{N}^{\pm\infty}, \max, +, -\infty, 0)$ and

$$\overset{2}{Ra} \qquad \overset{x+y}{\psi \wedge \varphi} \longrightarrow \overset{[x,y]}{[\psi, \varphi]} \qquad (x+y \longleftarrow [x,y])$$

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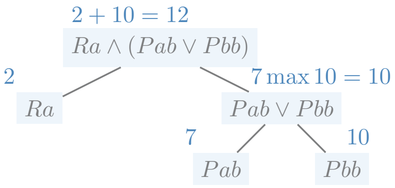

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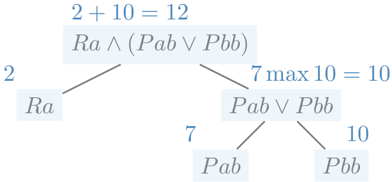

The *weight* of $a \in A$ is $[\![a]\!]$ with e.g. $[\![Ra \wedge (Pab \vee Pbb)]\!] = 12$.

Overview

Expressivity

Termination/Runtime

$\mathbb{S}_{\mathbb{N}^\infty} = (\mathbb{N}^\infty, +, \cdot, 0, 1)$

Safety

$\mathbb{S}_{\mathbb{B}} = (\{\text{false}, \text{true}\}, \vee, \wedge, \text{false}, \text{true})$

Combinations

$\mathbb{S} = \mathbb{S}_1 \times \mathbb{S}_2$

and more...

Prove Boundedness: $\forall a \in A : [\![a]\!] \prec \top$

Sufficient Conditions for Boundedness

e.g. $\mathsf{f}_{\mathrm{NF}} \leq C$ and all $\mathsf{Aggr}_{a \to B}$ selective

Proving via Interpretations

embedding $\mathfrak{e} : A \to \mathbb{S}$

Disproving via Increasing Loops

$a \xrightarrow{+1} b \xrightarrow{+1} a$

Techniques

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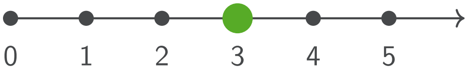

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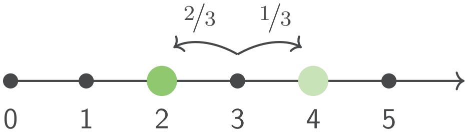

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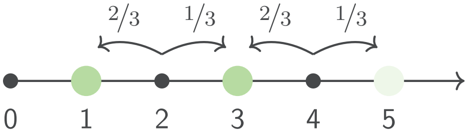

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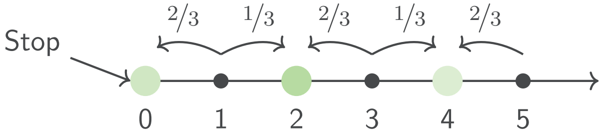

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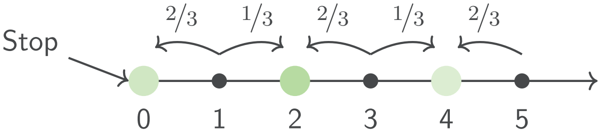
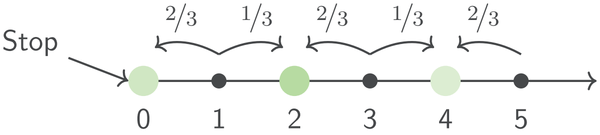


Sequence abstract reduction system $(\mathbb{N}, \to)$ with grammar

$$n + 1 \to [n, n + 2]$$

Complexity of the Random Walk



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and normal form $\mathtt{NF}_{\to} = \{0\}$.

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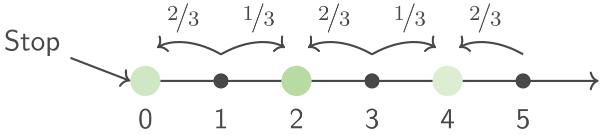

Sequence abstract reduction system $(\mathbb{N}, \to)$ with grammar

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and normal form $\mathtt{NF}_{\to} = \{0\}$.

What is the probability to reach 0 in 3 steps? What about finitely many steps?
What is the expected runtime?

Complexity of the Random Walk

Probability to reach 0 in 3 steps?

Complexity of the Random Walk

Probability to reach 0 in 3 steps? Choose $\mathbb{S}_{\mathbb{R}^\infty} = (\mathbb{R}^{\infty}_{\geq 0}, +, \cdot, 0, 1)$ and

Complexity of the Random Walk

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$$\boxed{0}$$

Complexity of the Random Walk

Probability to reach 0 in 3 steps? Choose $\mathbb{S}_{\mathbb{R}^\infty} = (\mathbb{R}^{\infty}_{\geq 0}, +, \cdot, 0, 1)$ and

$$\begin{array}{c} 1 \\ \boxed{0} \end{array}$$

Complexity of the Random Walk

Probability to reach 0 in 3 steps? Choose $\mathbb{S}_{\mathbb{R}^\infty} = (\mathbb{R}^{\infty}_{\geq 0}, +, \cdot, 0, 1)$ and

$$\overset{1}{\boxed{0}} \qquad \boxed{n+1}$$

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$$3$$

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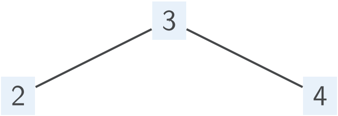

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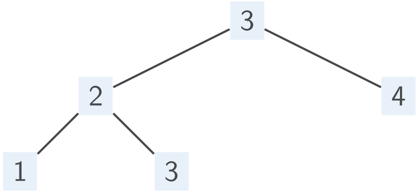

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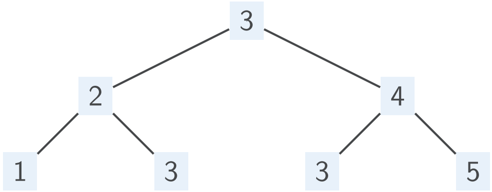

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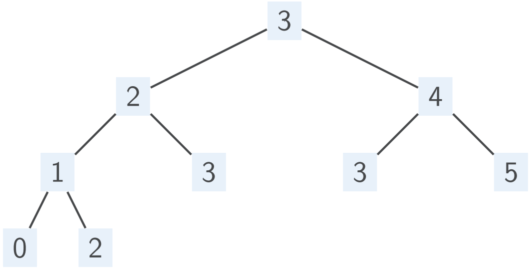

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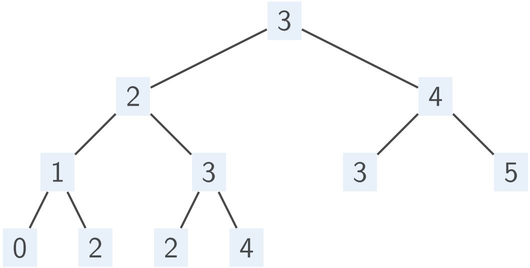

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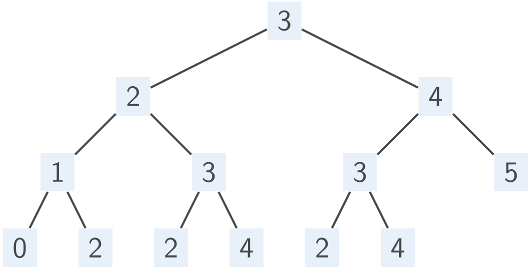

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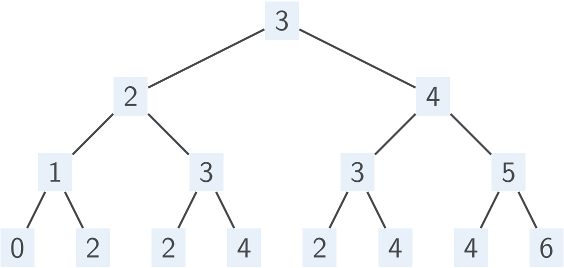

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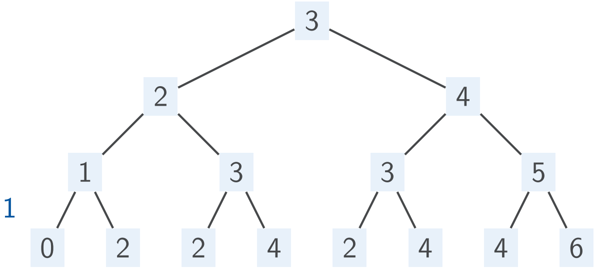

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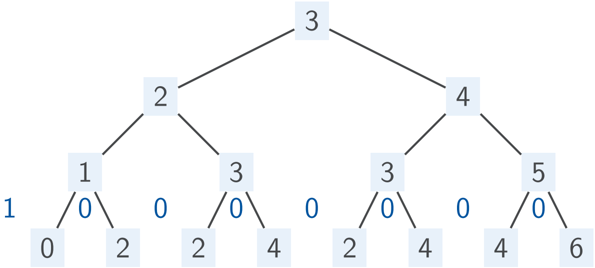

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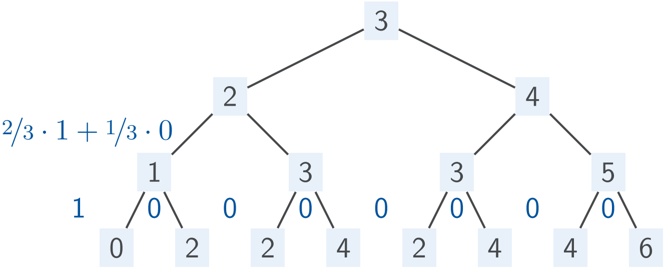

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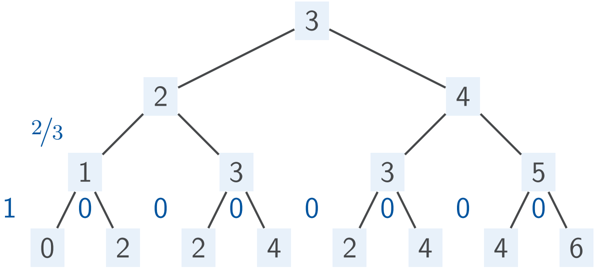

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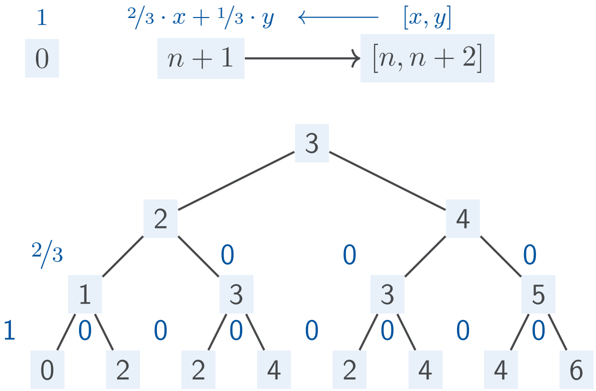

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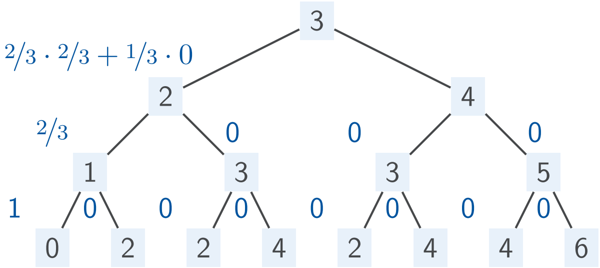

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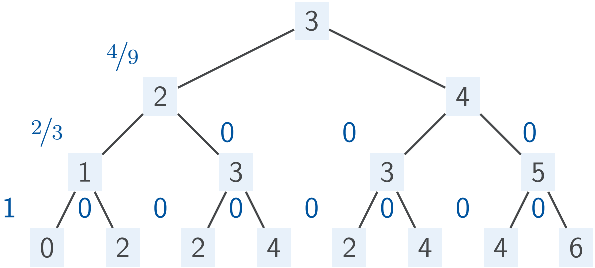

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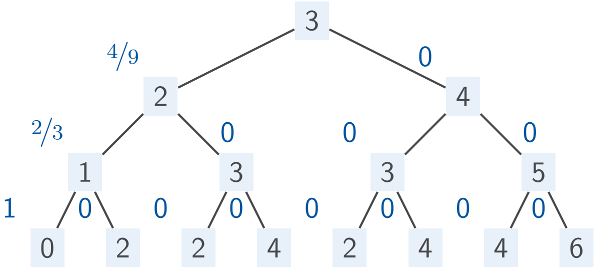

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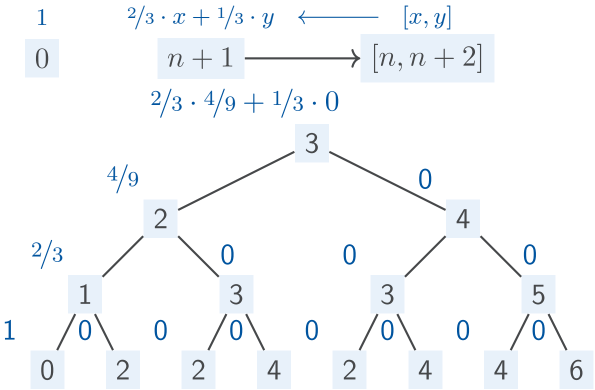

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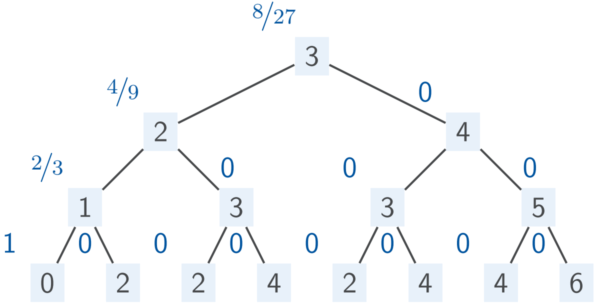

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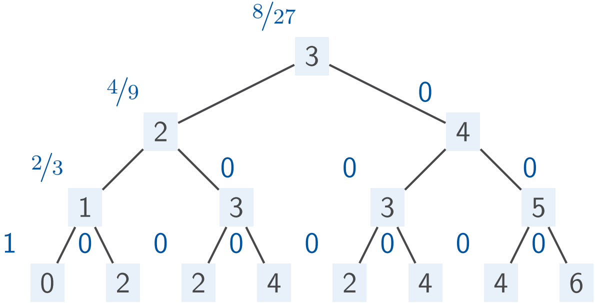

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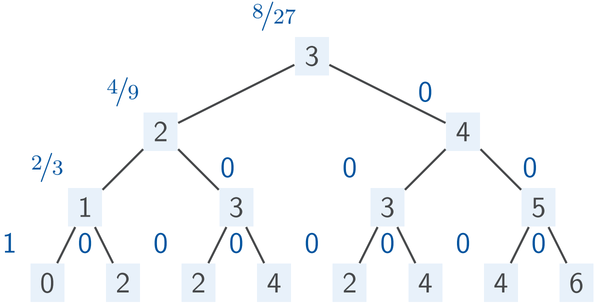
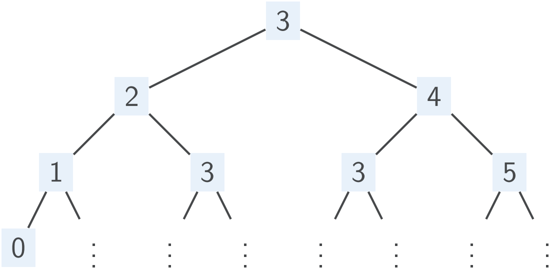


Probability to reach 0 $= \sup_{n \in \mathbb{N}} \{$ Probability to reach 0 in (at most) n steps $\}$ (Forward Unrolling)

Complexity of the Random Walk

What is the expected runtime?



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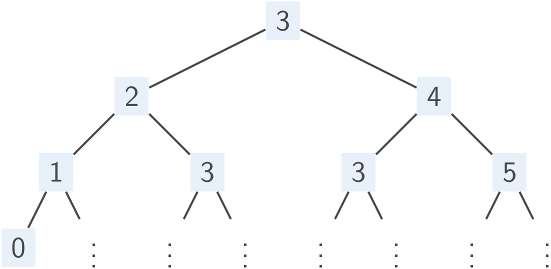

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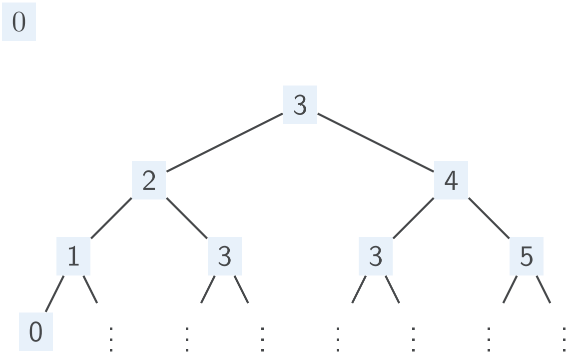

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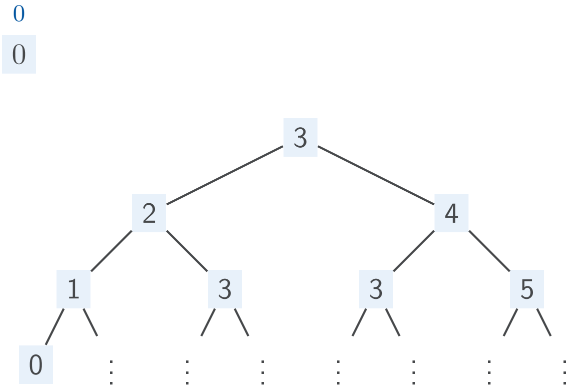

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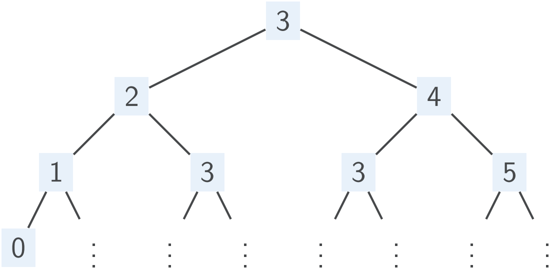
0

$\boxed{0}$ $\boxed{n+1}$



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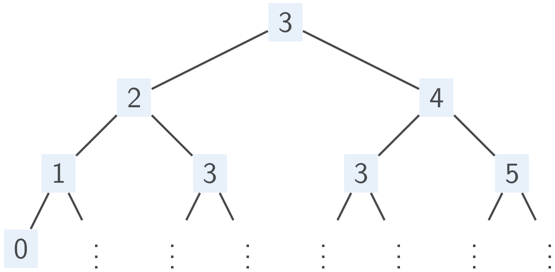

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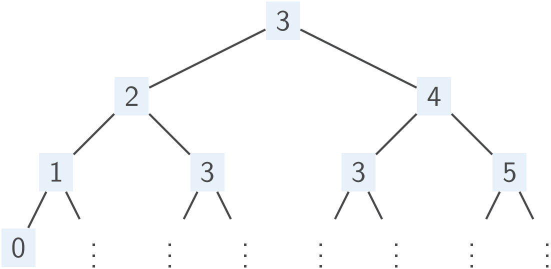

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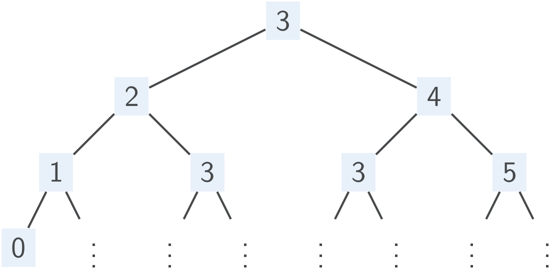

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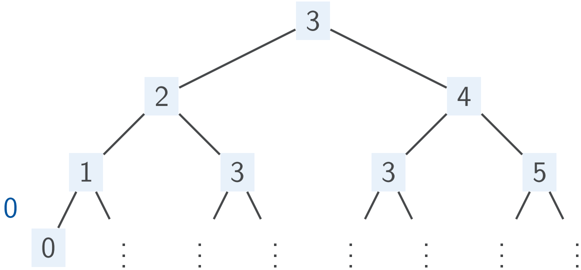

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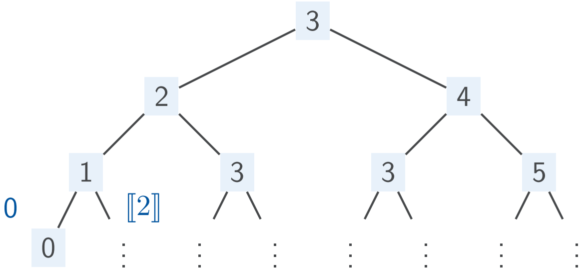

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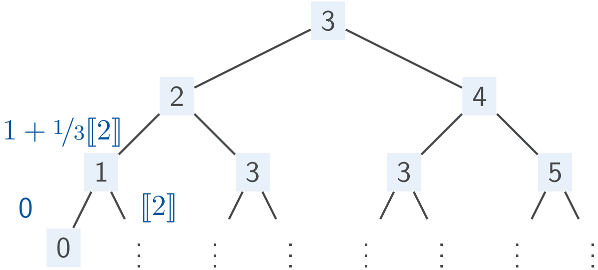

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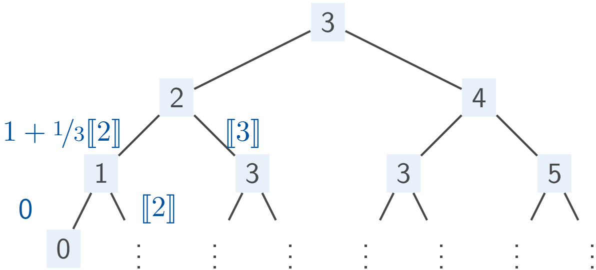

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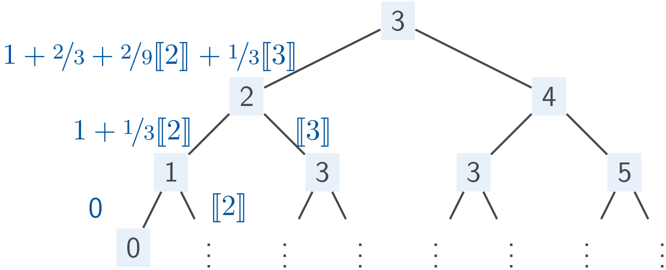

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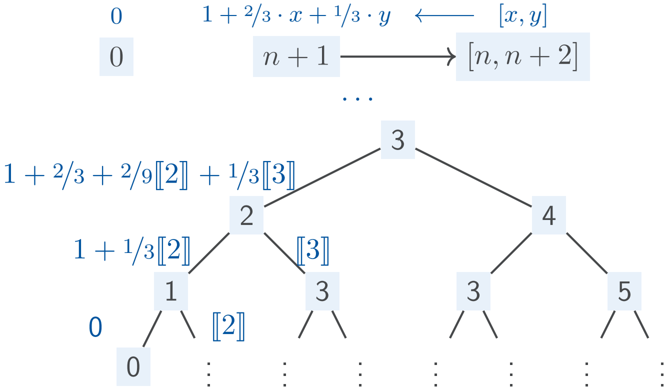
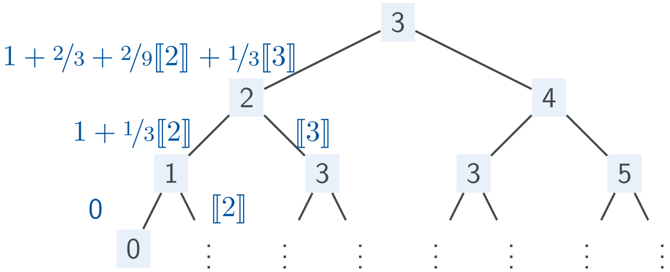


Complexity of the Random Walk

What is the expected runtime? Choose $\mathbb{S}_{\mathbb{R}^\infty} = (\mathbb{R}^\infty_{\geq 0}, +, \cdot, 0, 1)$ and

$$\begin{array}{ccc} 0 & 1 + {}^{2}/_{3} \cdot x + {}^{1}/_{3} \cdot y & \longleftarrow \quad [x, y] \\ \boxed{0} & \boxed{n+1} & \longrightarrow \quad \boxed{[n, n+2]} \end{array}$$

$\dots$

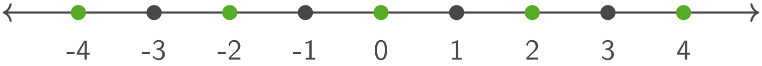


Is $[\![n]\!] \prec \infty$ for all $n \in \mathbb{N}$?

Termination and Runtime Complexity

Abstract reduction system $(\mathbb{Z}, \to)$ with

$$\begin{aligned} z \text{ is even and } z \leq -2: &\quad z \to z+2 \\ z \text{ is even and } z \geq 2: &\quad z \to z-2 \\ z \text{ is odd}: &\quad z \to -z \end{aligned}$$



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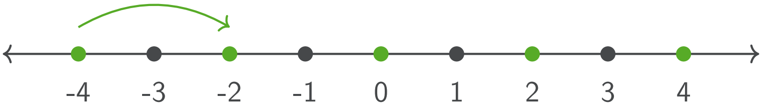

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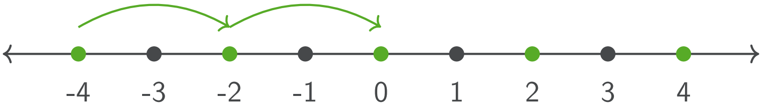

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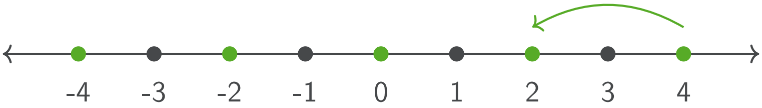

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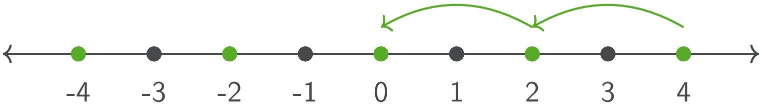

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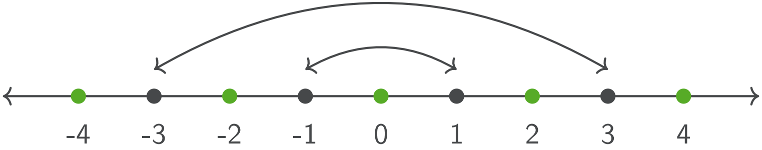

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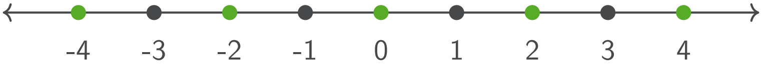

Do all reductions terminate?

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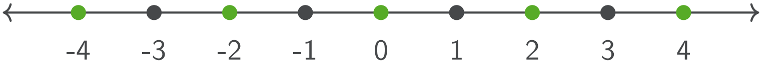

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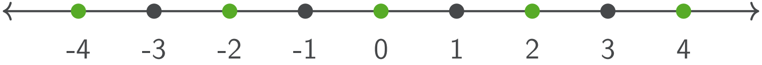

Do all reductions terminate? Choose $\mathbb{S}_{\mathbb{N}^\infty} = (\mathbb{N}, +, \cdot, 0, 1)$ and count steps. No!

Overview

Expressivity

Termination/Runtime

$\mathbb{S}_{\mathbb{N}^\infty} = (\mathbb{N}^\infty, +, \cdot, 0, 1)$

Safety

$\mathbb{S}_{\mathbb{B}} = (\{\mathsf{false}, \mathsf{true}\}, \vee, \wedge, \mathsf{false}, \mathsf{true})$

Combinations

$\mathbb{S} = \mathbb{S}_1 \times \mathbb{S}_2$

and more. . .

Prove Boundedness: $\forall a \in A : [\![a]\!] \prec \top$

Sufficient Conditions for Boundedness

e.g. $\mathsf{f}_{\mathrm{NF}} \leq C$ and all $\mathsf{Aggr}_{a \to B}$ selective

Proving via Interpretations

embedding $\mathfrak{e} : A \to \mathbb{S}$

Disproving via Increasing Loops

$a \xrightarrow{+1} b \xrightarrow{+1} a$

Techniques

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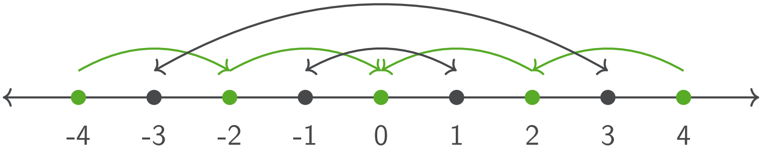

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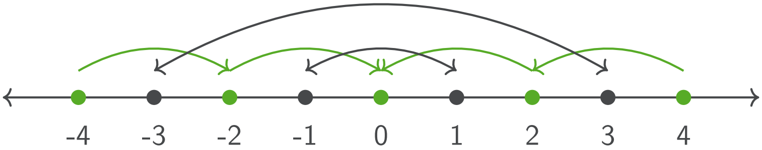

Hitting an even number is "unsafe". Are all reductions safe?

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$$\overset{\mathsf{true}}{\boxed{0}}$$

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$\boxed{4}$

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$$\begin{array}{c} \boxed{4} \\ | \\ \boxed{2} \\ | \\ \boxed{0} \end{array}$$

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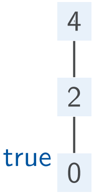

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$$\begin{array}{rc} & 4 \\ & | \\ \mathsf{true} & 2 \\ & | \\ \mathsf{true} & 0 \end{array}$$

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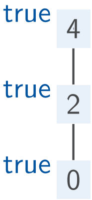

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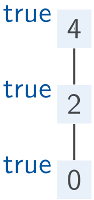

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true $\quad$ $\mathsf{true} \vee x \longleftarrow x$

$\boxed{0}$ $\qquad$ z is even: $\boxed{z} \longrightarrow \boxed{z \pm 2}$ $\qquad$ z is odd: $\boxed{z} \longrightarrow \boxed{-z}$

true $\boxed{4}$

true $\boxed{2}$

true $\boxed{0}$

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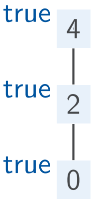

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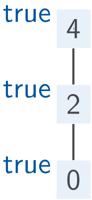

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$$\underset{\boxed{0}}{\mathsf{true}} \qquad z \text{ is even: } \boxed{z} \xrightarrow[\;\mathsf{true} \vee x \;\longleftarrow\; x\;]{} \boxed{z \pm 2} \qquad z \text{ is odd: } \boxed{z} \xrightarrow[\;\mathsf{false} \vee x \;\longleftarrow\; x\;]{} \boxed{-z}$$

$$\begin{array}{ll} \mathsf{true} & \boxed{4} \\ & \;| \\ \mathsf{true} & \boxed{2} \\ & \;| \\ \mathsf{true} & \boxed{0} \end{array} \qquad\qquad \boxed{5}$$

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true 4
|
true 2
|
true 0

5
|
-5
|
5

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$$\begin{array}{rc}
\mathsf{true} & 4 \\
 & | \\
\mathsf{true} & 2 \\
 & | \\
\mathsf{true} & 0
\end{array}
\qquad\qquad
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5 \\ | \\ -5 \\ | \\ 5 \\ | \\ \dots
\end{array}$$

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$$\begin{array}{rc}
\mathsf{true} & \boxed{4} \\
 & | \\
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 & | \\
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\end{array}
\qquad\qquad
\begin{array}{rc}
 & \boxed{5} \\
 & | \\
\mathsf{false} \vee [\![5]\!] & \boxed{-5} \\
 & | \\
[\![5]\!] & \boxed{5} \\
 & | \\
 & \ldots
\end{array}$$

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$$\begin{array}{rc} \mathsf{true} & 4 \\ & | \\ \mathsf{true} & 2 \\ & | \\ \mathsf{true} & 0 \end{array} \qquad\qquad \begin{array}{rc} \mathsf{false} \vee [\![5]\!] & 5 \\ & | \\ \mathsf{false} \vee [\![5]\!] & \text{-5} \\ & | \\ [\![5]\!] & 5 \\ & | \\ & \dots \end{array}$$

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$$\overset{\mathsf{true}}{4} \to \overset{\mathsf{true}}{2} \to \overset{\mathsf{true}}{0}$$

$$\overset{\mathsf{false} \vee [\![5]\!]}{5} \to \overset{\mathsf{false} \vee [\![5]\!]}{-5} \to \overset{[\![5]\!]}{5} \to \cdots$$

Is $[\![z]\!] < \mathsf{true}$ for all $z \in \mathbb{Z}$?

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true $\quad 0$

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z is odd: $\quad \mathsf{false} \vee x \longleftarrow x$ $\qquad z \longrightarrow -z$

true $\quad 4$

true $\quad 2$

true $\quad 0$

$\mathsf{false} \vee [\![5]\!]$ $\quad 5$

$\mathsf{false} \vee [\![5]\!]$ $\quad$ -5

$[\![5]\!]$ $\quad 5$

...

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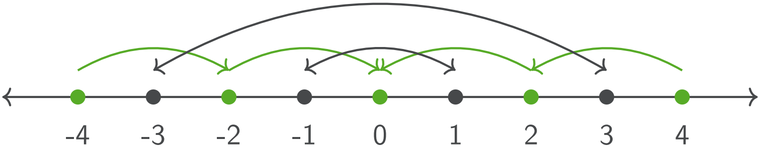
Combining Complexity and Safety

Abstract reduction system $(\mathbb{Z}, \to)$ with

$$\begin{aligned} z \text{ is even and } z \leq -2: &\quad z \to z+2 \\ z \text{ is even and } z \geq 2: &\quad z \to z-2 \\ z \text{ is odd}: &\quad z \to -z \end{aligned}$$

and normal form $\mathtt{NF}_{\to} = \{0\}$



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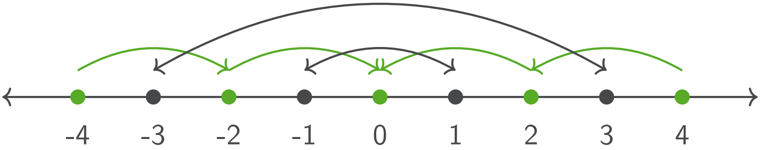

Are all runs terminating? Are all runs safe?

Combining Complexity and Safety

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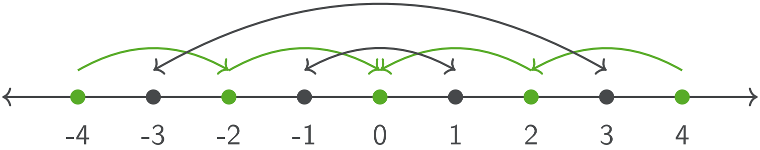


Are all runs terminating? Are all runs safe?

Now: Are all runs terminating or safe?

Combining Complexity and Safety

Are all runs terminating or safe?

Combining Complexity and Safety

Are all runs terminating or safe? Choose product semiring $\mathbb{S}_{\mathbb{N}^\infty} \times \mathbb{S}_{\mathbb{B}}$ and

z is even:

$$\begin{tikzcd} 4 \arrow[d, no head] \\ 2 \arrow[d, no head] \\ 0 \end{tikzcd}$$

z is odd:

$$\begin{tikzcd} 5 \arrow[d, no head] \\ -5 \arrow[d, no head] \\ 5 \arrow[d, no head] \\ \ldots \end{tikzcd}$$

Combining Complexity and Safety

Are all runs terminating or safe? Choose product semiring $\mathbb{S}_{\mathbb{N}^\infty} \times \mathbb{S}_{\mathbb{B}}$ and

0

z is even:

4
|
2
|
0

z is odd:

5
|
-5
|
5
|
...

Combining Complexity and Safety

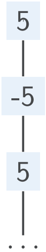
Are all runs terminating or safe? Choose product semiring $\mathbb{S}_{\mathbb{N}^\infty} \times \mathbb{S}_{\mathbb{B}}$ and

$(0, \mathsf{true})$

0

z is even:

z is odd:



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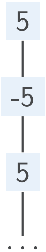

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Are all runs terminating or safe? Choose product semiring $\mathbb{S}_{\mathbb{N}^\infty} \times \mathbb{S}_{\mathbb{B}}$ and

$(0, \mathsf{true})$

0

z is even: $z \longrightarrow z \pm 2$ z is odd:

$$\begin{array}{c} 4 \\ | \\ 2 \\ | \\ 0 \end{array} \qquad\qquad \begin{array}{c} 5 \\ | \\ -5 \\ | \\ 5 \\ | \\ \cdots \end{array}$$

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Are all runs terminating or safe? Choose product semiring $\mathbb{S}_{\mathbb{N}^\infty} \times \mathbb{S}_{\mathbb{B}}$ and

$(0, \text{true})$

$\boxed{0}$ $\qquad z$ is even: $\qquad \boxed{z} \xrightarrow{x} \boxed{z \pm 2} \qquad z$ is odd:

4
|
2
|
0

5
|
-5
|
5
|
...

Combining Complexity and Safety

Are all runs terminating or safe? Choose product semiring $\mathbb{S}_{\mathbb{N}^\infty} \times \mathbb{S}_{\mathbb{B}}$ and

$$\overset{(0,\mathsf{true})}{0} \qquad z \text{ is even: } \quad z \xrightarrow{(1,\mathsf{true}) \oplus x \;\longleftarrow\; x} z \pm 2 \qquad z \text{ is odd:}$$

$$\begin{array}{c} 4 \\ | \\ 2 \\ | \\ 0 \end{array} \qquad\qquad \begin{array}{c} 5 \\ | \\ -5 \\ | \\ 5 \\ | \\ \ldots \end{array}$$

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Are all runs terminating or safe? Choose product semiring $\mathbb{S}_{\mathbb{N}^\infty} \times \mathbb{S}_{\mathbb{B}}$ and

$(0, \mathsf{true})$

$\boxed{0}$

z is even:

$$\boxed{z} \xrightarrow{(1, \mathsf{true}) \oplus x \;\longleftarrow\; x} \boxed{z \pm 2}$$

z is odd:

4

2

$(0, \mathsf{true})$ 0

5

-5

5

. . .

Combining Complexity and Safety

Are all runs terminating or safe? Choose product semiring $\mathbb{S}_{\mathbb{N}^\infty} \times \mathbb{S}_{\mathbb{B}}$ and

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$$\begin{array}{r c}
 & \boxed{4} \\
(1,\mathsf{true}) & | \\
 & \boxed{2} \\
(0,\mathsf{true}) & | \\
 & \boxed{0}
\end{array}
\qquad\qquad
\begin{array}{c}
\boxed{5} \\
| \\
\boxed{\text{-}5} \\
| \\
\boxed{5} \\
| \\
\dots
\end{array}$$

Combining Complexity and Safety

Are all runs terminating or safe? Choose product semiring $\mathbb{S}_{\mathbb{N}^\infty} \times \mathbb{S}_{\mathbb{B}}$ and

$\underset{\boxed{0}}{(0, \mathsf{true})}$

z is even: $\boxed{z} \xrightarrow{(1,\mathsf{true}) \oplus x \;\longleftarrow\; x} \boxed{z \pm 2}$ $\quad$ z is odd:

$$\begin{array}{rc}
(2, \mathsf{true}) & \boxed{4} \\
 & | \\
(1, \mathsf{true}) & \boxed{2} \\
 & | \\
(0, \mathsf{true}) & \boxed{0}
\end{array}
\qquad\qquad
\begin{array}{c}
\boxed{5} \\
| \\
\boxed{-5} \\
| \\
\boxed{5} \\
| \\
\dots
\end{array}$$

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Are all runs terminating or safe? Choose product semiring $\mathbb{S}_{\mathbb{N}^\infty} \times \mathbb{S}_{\mathbb{B}}$ and

$(0, \mathsf{true})$

0

z is even: $z \xrightarrow{(1, \mathsf{true}) \oplus x \;\longleftarrow\; x} z \pm 2$

z is odd: z

$(2, \mathsf{true})$ 4

$(1, \mathsf{true})$ 2

$(0, \mathsf{true})$ 0

5

-5

5

$\dots$

Combining Complexity and Safety

Are all runs terminating or safe? Choose product semiring $\mathbb{S}_{\mathbb{N}^\infty} \times \mathbb{S}_{\mathbb{B}}$ and

$$(0, \text{true}) \quad \boxed{0}$$

z is even: $$(1, \text{true}) \oplus x \quad \longleftarrow \quad x$$
$$\boxed{z} \longrightarrow \boxed{z \pm 2}$$

z is odd: $$\boxed{z} \longrightarrow \boxed{-z}$$

$(2, \text{true})$ 4
$(1, \text{true})$ 2
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5
-5
5
...

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z is odd: $\boxed{z} \xrightarrow{x} \boxed{-z}$

$(2, \mathsf{true})$ $\boxed{4}$

$(1, \mathsf{true})$ $\boxed{2}$

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$\boxed{5}$

$\boxed{-5}$

$\boxed{5}$

$\ldots$

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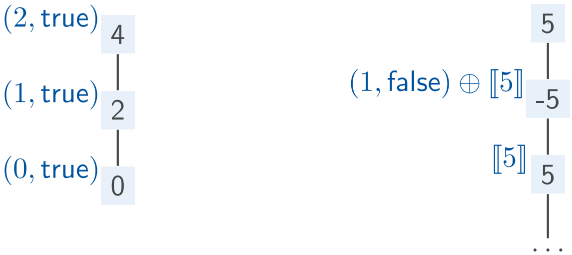

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$\boxed{0}$

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$(2, \text{true})$ 4

$(1, \text{true})$ 2

$(0, \text{true})$ 0

$(2, \text{false}) \oplus [\![5]\!]$ 5

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$[\![5]\!]$ 5

$\ldots$

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Is $[\![z]\!] < (\infty, \mathsf{true})$ for all $z \in \mathbb{Z}$?

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$$\boxed{0} \quad (0, \mathsf{true})$$

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Is $[\![z]\!] < (\infty, \mathsf{true})$ for all $z \in \mathbb{Z}$? Yes!

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Random Walk: Choose $\mathbb{S}_{\mathbb{R}^\infty} = (\mathbb{R}^{\infty}_{\geq 0}, +, \cdot, 0, 1)$ and

$$\begin{array}{ccccc}
0 & \qquad & 1 + {}^{2}\!/_{3} \cdot x + {}^{1}\!/_{3} \cdot y & \longleftarrow & [x, y] \\
\boxed{0} & \qquad & \boxed{n+1} & \longrightarrow & \boxed{[n, n+2]}
\end{array}$$

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Choose $\mathfrak{e} : n \mapsto 3 \cdot n$ such that

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$$\begin{array}{ccccc} 0 & \qquad & 1 + {}^{2}/_{3} \cdot x + {}^{1}/_{3} \cdot y & \longleftarrow & [\mathfrak{e}(n), \mathfrak{e}(n+2)] \\ \boxed{0} & \qquad & \boxed{n+1} & \longrightarrow & \boxed{[n, n+2]} \end{array}$$

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$$\begin{array}{ccccc} 0 & \qquad & 1 + {}^{2}\!/_{3} \cdot x + {}^{1}\!/_{3} \cdot y & \longleftarrow & [3n, 3n+6] \\ \boxed{0} & & \boxed{n+1} & \longrightarrow & \boxed{[n, n+2]} \end{array}$$

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$$\begin{array}{ccccc} 0 & \qquad & 1 + {}^{2}\!/_{3} \cdot 3n + {}^{1}\!/_{3} \cdot (3n+6) & \leftarrow & [3n, 3n+6] \\ \boxed{0} & & \boxed{n+1} & \longrightarrow & \boxed{[n, n+2]} \end{array}$$

Choose $\mathfrak{e} : n \mapsto 3 \cdot n$ such that

- $\mathfrak{e}(a) \succeq [\![a]\!] = \mathsf{f}_{\mathtt{NF}}(a)$ for all normal forms $a \in A$.
- $\mathfrak{e}(a) \succeq \mathsf{Aggr}_{a \to [b_1, \ldots, b_n]}(\mathfrak{e}(b_1), \ldots, \mathfrak{e}(b_n))$ for all $a \to [b_1, \ldots, b_n]$.

Proving via Interpretations

$(A, \to, \mathbb{S}, \mathsf{f}_{\mathtt{NF}}, \mathsf{Aggr})$ is bounded $:\Leftrightarrow$ for all $a \in A$: $[\![a]\!] \prec \top$

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Random Walk: Choose $\mathbb{S}_{\mathbb{R}^\infty} = (\mathbb{R}^{\infty}_{\geq 0}, +, \cdot, 0, 1)$ and

$$
\begin{array}{ccccc}
0 & \qquad & 1 + 2n + n + 2 & \longleftarrow & [3n, 3n+6] \\
\boxed{0} & \qquad & \boxed{n+1} & \longrightarrow & \boxed{[n, n+2]}
\end{array}
$$

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$$\begin{tikzcd}
0 & 3n+3 & {[3n, 3n+6]} \arrow[l] \\
0 & \mathfrak{e}(n+1) \arrow[r] & {[n, n+2]}
\end{tikzcd}$$

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Is the expected runtime finite? Yes, $[\![n]\!] \leq 3 \cdot n \prec \infty$!

Overview

Expressivity

Termination/Runtime

$\mathbb{S}_{\mathbb{N}^\infty} = (\mathbb{N}^\infty, +, \cdot, 0, 1)$

Safety

$\mathbb{S}_{\mathbb{B}} = (\{\mathsf{false}, \mathsf{true}\}, \vee, \wedge, \mathsf{false}, \mathsf{true})$

Combinations

$\mathbb{S} = \mathbb{S}_1 \times \mathbb{S}_2$

and more...

Prove Boundedness: $\forall a \in A : [\![a]\!] \prec \top$

Sufficient Conditions for Boundedness

e.g. $\mathsf{f}_{\mathtt{NF}} \leq C$ and all $\mathsf{Aggr}_{a \to B}$ selective

Proving via Interpretations

embedding $\mathfrak{e} : A \to \mathbb{S}$

Disproving via Increasing Loops

$a \xrightarrow{+1} b \xrightarrow{+1} a$

Techniques

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$a \xrightarrow{+1} b \xrightarrow{+1} a$

Techniques

Disproving via Increasing Loops

$(A, \to, \mathbb{S}, \mathrm{f}_{\mathtt{NF}}, \mathrm{Aggr})$ is unbounded $:\Leftrightarrow$ there exists an $a \in A$: $[\![a]\!] = \top$

- Show unboundedness via induced weight polynomial $\mathcal{P}_a(X)$ of loops $a \to^* a$

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Termination: Choose $\mathbb{S}_{\mathbb{N}^\infty} = (\mathbb{N}^\infty, +, \cdot, 0, 1)$ and

$$
\begin{array}{ccccccccc}
0 & \qquad & 1 + x & \longleftarrow & x & \qquad & 1 + x & \longleftarrow & x \\
\boxed{0} & & \boxed{z} & \longrightarrow & \boxed{-z} & & \boxed{z} & \longrightarrow & \boxed{z \pm 2}
\end{array}
$$

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$$\begin{array}{ccc} 0 & 1 + x \longleftarrow x & 1 + x \longleftarrow x \\ \boxed{0} & \boxed{z} \longrightarrow \boxed{-z} & \boxed{z} \longrightarrow \boxed{z \pm 2} \end{array}$$

$\boxed{5}$

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$$\begin{array}{c} 0 \\ 0 \end{array} \qquad \begin{array}{ccc} 1 + x & \longleftarrow & x \\ z & \longrightarrow & -z \end{array} \qquad \begin{array}{ccc} 1 + x & \longleftarrow & x \\ z & \longrightarrow & z \pm 2 \end{array}$$

$$5 \longrightarrow -5$$

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$$\boxed{5} \longrightarrow \boxed{-5} \longrightarrow \boxed{5}$$

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$$\overset{0}{\boxed{0}} \qquad \boxed{z} \xrightarrow{1+x \,\longleftarrow\, x} \boxed{-z} \qquad \boxed{z} \xrightarrow{1+x \,\longleftarrow\, x} \boxed{z \pm 2}$$

$$\boxed{5} \longrightarrow \boxed{-5} \longrightarrow \overset{X}{\boxed{5}}$$

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$$\boxed{5} \longrightarrow \boxed{-5} \longrightarrow \boxed{5} \qquad 1 + X \longleftarrow X$$

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Are all reductions terminating? No!

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$$\mathcal{P}_5(X) = 2 + X, \qquad \bigoplus_{i\in\mathbb{N}} 2 = \infty \checkmark$$

$$\mathcal{P}_5(X) = \max\{1, X\}, \qquad \bigoplus_{i\in\mathbb{N}} 1 = 1 \text{ ✗}$$

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Techniques

One Framework Fits All

- Framework to unify research questions
- Lifts techniques from one area to another
- Future Work:
  - Lift more techniques to the general setting
  - Increase expressivity further
    - Starvation Freedom
  - Automate the techniques

Abstract Reduction System

$(A, \to)$

Provenance Analysis via Semirings

$\mathbb{S}_{\mathbb{N}^\infty} = (\mathbb{N}^\infty, +, \cdot, 0, 1)$

Weighted Rewriting

Uniform Framework for Analyzing Abstract Reduction Systems

- Termination Analysis
- Safety Analysis
- Cost Analysis
- Complexity Analysis
- Arbitrary Combinations
- Many more!

Emma Ahrens, Jan-Christoph Kassing, Jürgen Giesl, Joost-Pieter Katoen:
Weighted Rewriting: Semiring Semantics for Abstract Reduction Systems.